THE NUMBER *e*

Recall the special number π. It is the ratio of the circumference of any circle to its diameter, and we learned that it's irrational (an infinite, nonrepeating decimal). There's another extremely important irrational number and it's the topic of this chapter. It will be defined in terms of a limit, and will be the basis of applications in banking, biology, and radioactivity. Moreover, in Calculus it's absolutely the most important base for exponential functions.

❑ NEW LIMIT NOTATION

Let's review an example of limits. Consider the function

$$f(x) = \frac{1}{x}.$$

We know that as x grows larger and larger, $f(x)$ gets closer and closer to 0. We usually write:

$$\text{As } x \to \infty,\ f(x) \to 0.$$

In our new notation, we would write this fact as

$$\lim_{x\to\infty} f(x) = 0.$$

In general, instead of writing

$$\text{As } x \to a, f(x) \to L,$$

we write

$$\lim_{x\to a} f(x) = L.$$

These two statements mean exactly the same thing.

Homework

1. If $h(x) = x^2$, find $\lim_{x \to 3} h(x)$.

2. If $f(x) = \dfrac{2x+1}{x-3}$, find $\lim_{x \to \infty} f(x)$. Hint: Consider the graph of f.

3. If $E(x) = \dfrac{9}{1+x^2}$, find $\lim_{x \to 0} E(x)$.

❑ *Earning Interest at the Bank*

In **simple interest** at the bank, the interest you earn on your investment is calculated just once at the end of the year. Suppose you place \$70 in the bank at 10% interest per year. At the end of the year you will have earned \$7 in interest, making your current balance \$77.

Now let's consider **compound interest**. As an example, we'll divide the year into four compounding periods (each one called a *quarter*), and we'll pretend we have \$1 invested at 100% annual interest. At the end of each quarter, you earn interest in the amount of 25% (100% divided into 4 equal parts) of your current balance. This means that from now on, your interest itself is earning interest.

Start: \$1

1st Quarter: previous balance + 25% of the previous balance

= \$1 + 25% of \$1

= $\$1 + \$\frac{1}{4}$ (balance = \$1.25 after 1 quarter)

2nd Quarter: previous balance + 25% of the previous balance

= $(\$1 + \$\frac{1}{4}) + 25\%$ of $(\$1 + \$\frac{1}{4})$

$$= (\$1 + \$\tfrac{1}{4}) + \tfrac{1}{4}(\$1 + \$\tfrac{1}{4})$$

$$= (\$1 + \$\tfrac{1}{4})(\$1 + \$\tfrac{1}{4}) \qquad \text{(factor out } 1 + \tfrac{1}{4}\text{)}$$

$$= (\$1 + \$\tfrac{1}{4})^2 \qquad \text{(balance = \$1.56 after 2 quarters)}$$

<u>3rd Quarter:</u> previous balance + 25% of the previous balance

$$= (\$1 + \$\tfrac{1}{4})^2 + 25\% \text{ of } (\$1 + \$\tfrac{1}{4})^2$$

$$= (\$1 + \$\tfrac{1}{4})^2 + \tfrac{1}{4}(\$1 + \$\tfrac{1}{4})^2$$

$$= (\$1 + \$\tfrac{1}{4})^2(\$1 + \$\tfrac{1}{4}) \qquad \text{(factor out } (1 + \tfrac{1}{4})^2\text{)}$$

$$= (\$1 + \$\tfrac{1}{4})^3 \qquad \text{(balance = \$1.95 after 3 quarters)}$$

<u>4th Quarter:</u> previous balance + 25% of the previous balance

$$= (\$1 + \$\tfrac{1}{4})^3 + 25\% \text{ of } (\$1 + \$\tfrac{1}{4})^3$$

$$= (\$1 + \$\tfrac{1}{4})^3 + \tfrac{1}{4}(\$1 + \$\tfrac{1}{4})^3$$

$$= (\$1 + \$\tfrac{1}{4})^3(\$1 + \$\tfrac{1}{4}) \qquad \text{(factor out } (1 + \tfrac{1}{4})^3\text{)}$$

$$= (\$1 + \$\tfrac{1}{4})^4 \qquad \text{(balance = \$2.44 after 1 full year)}$$

<u>Note:</u> To compare simple interest with compound interest, \$1.00 invested for one year at 100% <u>simple</u> interest would yield \$1.00 in interest, for a final balance of **\$2.00**. Using <u>compound</u> interest, we see that the last calculation in the 4th quarter above is

$$(\$1 + \$\tfrac{1}{4})^4 = \$1.25^4 = \mathbf{\$2.44}$$

To summarize, investing \$1 at 100% annual interest compounded four times a year yields a final balance, after one year in the bank, of

$$(\$1 + \$\tfrac{1}{4})^4$$

From this example we can generalize as follows: Investing \$1 at 100% annual interest compounded ***n*** times a year yields a final balance of

$$\left(\$1 + \$\frac{1}{\boldsymbol{n}}\right)^{\boldsymbol{n}}$$

at the end of one year. For instance, investing $1 at 100% annual interest compounded 360 times a year would give you a balance of

$$\left(1+\frac{1}{360}\right)^{360} = (1.002777778)^{360} = \mathbf{\$2.7145}$$

Summarizing,

> Investing $1 at 100% annual interest for one year, compounded *n* times a year, yields a final balance of
>
> $$\left(1+\frac{1}{n}\right)^{n}$$

Homework

4. For each problem, use the formula $\left(1+\frac{1}{n}\right)^{n}$ to find the balance at the end of one year if the original investment was $1, earning 100% annual interest, and assuming the given number of compounding periods:

   a. 1 b. 12 c. 365 d. 1,000 e. 5,000,000

❑ *THE MOST IMPORTANT EXPONENTIAL BASE OF ALL !*

Now, suppose that we consider compounding so frequently that it occurs at every instant of time. This is called ***continuous compounding***. What will your account balance be after one year of continuous compounding? It would come from the formula

$$\left(1+\frac{1}{n}\right)^n$$

with n replaced by ∞: $\left(1+\frac{1}{\infty}\right)^{\infty}$. But this would be meaningless. So we use the new limit notation introduced a few pages back. We keep the n's in the formula, and then specify that n should approach infinity; that is, n becomes infinitely large:

> Whatever every instant of time means, it drove the Greeks bonkers, and it wasn't until the 17th century that Newton and Leibniz made some sense of this when they invented Calculus.]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It's hard to see this now, but the limit expression above is actually a specific real number. In fact, this number is one of the most important numbers in math, science, statistics, and business. It is given the name "***e***", perhaps due to the word "*e*xponential," or perhaps due to its creator Leonard *E*uler.

Though ***e*** is defined as a limit, a decimal approximation of ***e*** will be discussed in the homework. In summary,

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

Homework

5. Define e in two ways: as a limit, and as the result of a certain investment.

6. The number e is irrational, although I have no idea how to prove it. Nevertheless, explain exactly what it means for e to be irrational.

7. a. Use the definition of e with a value of n = 10,000,000 to approximate the value of e.

   b. Use the e^x button on your calculator to approximate e. Hint: $e = e^1$.

   c. Which approximation do you think is more accurate?

8. Use your calculator to approximate each of the following:

   a. $e^{3.7}$ b. $\dfrac{1}{e}$ c. $\sqrt[7]{e}$ d. $\dfrac{e^2 - \sqrt{e}}{\sqrt[3]{4e+1}}$

9. Some books define $\boldsymbol{e}$ as $\displaystyle\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$. Prove that this is the same as our definition.

❑ *The Laws of Exponents*

All the laws for exponents with which we're familiar work for exponential expressions like 2^x and e^x just as well as they did for x^2. Here are the six main rules. Assume a and b are positive constants.

$$a^{-x} = \frac{1}{a^x} \qquad a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Homework

10. Simplify each expression:

a. $2^x 2^y$ b. e^0 c. e^{-5} d. $\dfrac{e^5}{e^2}$ e. $\dfrac{e^x}{e^y}$

f. $(e^x)^e$ g. $(10e)^x$ h. $\left(\dfrac{e}{2}\right)^n$ i. $(e^\pi)^{10x}$ j. $2^3 2^0$

Review Problems

11. By making a sketch of the function $y = \dfrac{2}{x}$, calculate:

a. $\lim\limits_{x \to \infty} y$ b. $\lim\limits_{x \to -\infty} y$

12. a. Define e as it applies to continuous compounding in the bank.
b. Define e as a limit.
c. Is e a real number?
d. Is e rational or irrational?
e. Explain exactly why e is a valid base for an exponential function.
f. Is $e > \pi$ or is $e < \pi$?

13. a. Simplify: $e^0 + \dfrac{e^7}{e^2} + (e^3)^4$ b. Approximate: $e^3 + \sqrt[3]{e}$
c. Approximate: $(\pi - e)^2$ d. Approximate: e^π.

Solutions

1. In this limit, x is approaching 3. This means that x is approaching 3 by taking values like 3.2, 3.1, 3.05, 3.001, 3.000001 – or perhaps x is taking values like 2.9, 2.95. 2.99, 2.999, etc. Now, as x does this, what does the functional value x^2 do? Well, it gets closer and closer to 9. Therefore, the limit is 9.

2. Since x is approaching infinity, this limit is equivalent to finding a horizontal asymptote. Take your calculator and evaluate the function for a really huge value of x. You should see that the limit is 2.

3. As x shrinks toward 0, the x^2 term in the denominator essentially disappears, leaving a functional value of 9, which is the answer to the limit question.

4. a. \$2.00 b. \$2.6130 c. \$2.7146 d. \$2.7169 e. \$2.7183

5. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$

If you invest \$1 at a 100% interest rate compounded continuously, your balance at the end of one year will be \$$e$.

6. e is an infinite, non-repeating decimal.

7. a. My TI-30X gives the answer 2.718288827, but your calculator might give 2.718281693, or something close.

b. 2.718281828 c. The second value

8. a. 40.4473 b. 0.3679 c. 1.1536 d. 2.5162

9. Since $1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$, the definitions are equivalent.

10. a. 2^{x+y} b. 1 c. $\frac{1}{e^5}$ d. e^3 e. e^{x-y}

f. e^{ex} g. $10^x e^x$ h. $\frac{e^n}{2^n}$ i. $e^{10\pi x}$ j. 8

11. a. 0 b. 0

12. a. Invest \$1 for 1 year at an interest rate of 100%/yr compounded <u>continuously</u>. At the end of the year, the account balance is \$*e*.

b. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ c. Yes d. Irrational

e. *e* is real number that is greater than 0 but not equal to 1.

f. $e < \pi$

13. a. $1 + e^5 + e^{12}$ b. 21.4811 c. 0.1792 d. 23.1407

“Many of life’s failures are people who did not realize how close they were to success when they gave up.”

– Thomas Edison